INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust Nonlinear Control (2016)
Published online in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.3518

Fault tolerant control design using adaptive control allocation based on the pseudo inverse along the null space

S. S. Tohidi[1,2], A. Khaki Sedigh[1,2,*,†] and D. Buzorgnia[2]

[1]*Center of Excellence in Industrial Control, Department of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran*
[2]*APAC Research Group, K. N. Toosi University of Technology, Tehran, Iran*

SUMMARY

Fault-tolerant control systems are vital in many industrial systems. Actuator redundancy is employed in advanced control strategies to increase system maneuverability, flexibility, safety, and fault tolerability. Management of control signals among redundant actuators is the task of control allocation algorithms. Simplicity, accuracy and low computational cost are key issues in control allocation implementations. In this paper, an adaptive control allocation method based on the pseudo inverse along the null space of the control matrix (PAN) is introduced in order to adaptively tolerate actuator faults. The proposed method solves the control allocation problem with an exact solution and optimized l_∞ norm of the control signal. This method also handles input limitations and is computationally efficient. Simulation results are used to show the effectiveness of the proposed method.

Received 8 November 2013; Revised 30 November 2015; Accepted 22 December 2015

KEY WORDS: fault-tolerant control; control allocation; adaptive control; overactuated systems.

1. INTRODUCTION

Hardware redundancy, for example, actuator or sensor redundancy, is commonly used in industrial applications to increase system maneuverability, flexibility, safety, and fault tolerability [1–3]. However, actuator redundancies lead to system complexity, and designing controllers become more difficult. To design and implement simple control strategies, control allocation algorithms can be used to manage control signals among redundant actuators using the degrees of freedom provided by the actuator redundancies.

There are various control allocation methodologies available in the literature [4, 5]. The constrained least squares and its modifications [5, 6] are computationally simple approaches to control allocation. Daisy chaining is another method for managing control signals among redundant actuators [7]. In this method, the actuators are divided into certain groups, and the system uses these groups sequentially when needed. Energy-consumption reduction is a key advantage of the daisy chaining method. Direct allocation is a common control allocation method that is based on the pseudo inverse concept and can consider control signal constraints [5, 8]. This method is transformed into a constrained optimization problem in [9]. Many control allocation methods are based on optimization problems. Optimization methods have high computational cost. Error minimization using linear programming [4] minimizes the weighted error between the allocated virtual control and the desired control, which is solved using iterative numerical algorithms [9, 10]. Simplex method, active set method, and interior point method are used to solve the linear programming problem

*Correspondence to: A. Khaki Sedigh, Center of Excellence in Industrial Control, Department of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran.
†E-mail: asedigh@yahoo.com

[9, 11]. Using quadratic programming is another optimization-based solution for the control allocation problem [11–13]. Also, nonlinear programming methods are used to solve the control allocation problem [14, 15]. In recent years, much research has been performed on dynamic allocators. In comparison with the traditional view that considers control allocation as a mapping from virtual control to system's input, these methods are based on differential equations that dynamically show the behavior of control allocation. In [16], dynamic input allocation is proposed based on two redundancy definitions, where anti-windup compensators are used to consider actuators position and rate constraints. In [17], a dynamic input allocation is proposed, which consists of an optimal steady state input allocation and an annihilator, which guarantees output invisibility of input allocator design. In [18], dynamic allocation is used to design output regulators in overactuated systems.

Fault-tolerant control is an active research area. Two main fault-tolerant methodologies are the active and passive methods [1, 19, 20]. The passive methods are based on robust-fixed-structure control systems considering bounds of uncertainty [21–23]. As the number of faults and degree of redundancy increases, the controller design based on the passive approach becomes more conservative [24]. Control allocation is an appealing approach to the design of active fault-tolerant control systems [1]. In modular systems using control allocation, controller is designed based on the system in normal situation. The important property of control allocation methods is using actuator redundancies and managing the control signals among actuators with no need to change controllers in faulty situations. Adaptive control allocation methods are used in order to accommodate control allocation based on the changes in system parameters [25–28].

Faults change system dynamics and controlling systems in faulty situations is a key issue in practical systems. To control systems facing faults, it is necessary to adaptively identify the fault effects. One method is to use fault detection and isolation algorithms (FDI) and based on the type and place of faults, the designed controller decreases the effect of faults [29]. In [30], a family of independent unknown input observers are proposed to isolate faults in overactuated systems. Another method is to use adaptive fault identifiers in order to make decisions based on the faults on the systems [31].

This paper proposes a new control allocation method that can adaptively tolerate faults in systems with actuator redundancies. This method is based on modifying the pseudo inverse approach along the null space of the control matrix [32]. Simplicity, accuracy, low computational cost, and handling input limitations are the main characteristics of the proposed control allocation method.

This paper is organized as follows. Section 2 presents the problem. Model of the system and fault are presented in this section. Section 3 presents the proposed control allocation method. An algorithmic approach to constraint consideration is described in this section. Also, solving infeasibility and singularity problems are two important issues in this section. In section 4, two examples are used to show the main points of the proposed method. These include a simple scalar system and a supply vessel model in a realistic environment to illustrate the effect of fault-tolerant adaptive control allocation method. Also, comparison results are provided in this section. Finally, section 5 concludes the paper.

2. PROBLEM STATEMENTS

Consider the plant described by the following discrete-time state space equations

$$x(t+1) = Ax(t) + Bu(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the system states and control inputs. It is assumed that the system has redundant actuators and therefore the control matrix is rank deficient

$$rank(B^{n\times m}) = d < m \tag{2}$$

and also the control signals are constrained as

DOI: 10.1002/rnc

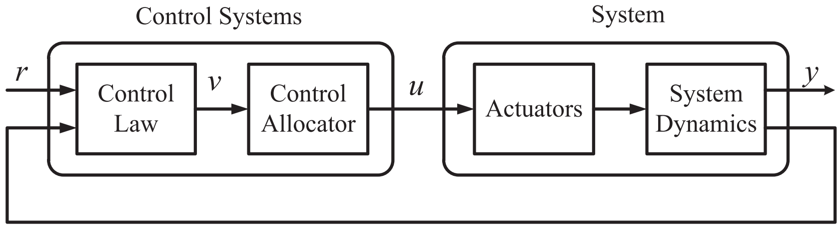

Figure 1. Modular structure using control allocation.

$$u(t) \in \Omega \equiv \{u \in \mathbb{R}^m | u^- \leqslant u(t) \leqslant u^+\}, \tag{3}$$

where the constraints are defined so that $u^- \equiv [u_1^-, u_2^-, \ldots, u_m^-]^T$ and $u^+ \equiv [u_1^+, u_2^+, \ldots, u_m^+]^T$ [27].

It is now assumed that the system is subject to actuator faults and can be written as [25]

$$x(t+1) = Ax(t) + Bu(t) - BK(t)u(t), \tag{4}$$

where $k_1(t), \ldots, k_m(t)$ are the effectiveness gains of actuators. By defining $K(t)$ as

$$K(t) = diag(k_1(t), \ldots, k_m(t)), \qquad 0 \leqslant k_i(t) \leqslant 1 \tag{5}$$

the state space equation of the system facing actuator faults are given as

$$\begin{aligned} x(t+1) &= Ax(t) + BW(t)u(t) \\ W(t) &= I_m - K(t) \end{aligned}, \tag{6}$$

where $W(t) = diag(w_1(t), \ldots, w_m(t))$ is the effectiveness matrix. If $k_i(t) = 0$, the ith actuator is working perfectly, and if $k_i(t) > 0$, the ith actuator is faulty, and if $k_i(t) = 1$, the ith actuator has completely failed.

3. THE PROPOSED CONTROL ALLOCATION METHODOLOGY

3.1. Pseudo inverse along the null space

Control allocation strategies are required to manage the control signals among the redundant actuators. A practical control allocation methodology should be optimal, handle the input constraints, provide accurate results, and be computationally efficient.

Consider the following non-faulty overactuated system

$$\begin{aligned} x(t+1) &= Ax(t) + v(t) \\ v(t) &= Bu(t), \quad u^- \leqslant u(t) \leqslant u^+ \end{aligned}, \tag{7}$$

where $v(t)$ is the virtual control signal produced by the control law as it is seen in Figure 1. The fault-tolerant control system proposed in this paper is based on the control allocation methods with actuator redundancies. When faults occur in the actuators, the system can be modeled as (6), and by considering $BW(t) = B_f(t)$, the virtual control in faulty situation is defined as

$$v_f(t) = BW(t)u(t) \Longrightarrow v_f(t) = B_f(t)u(t) \tag{8}$$

$$rank(BW(t)) = rank(B_f(t)) = k' < m. \tag{9}$$

If (8) has a solution within the system actuator constraints, it is feasible, but if it has no solution within the defined constraints, there should be a methodology that leads to solutions within the constraints. For simplicity, the time index is omitted, for example, using u instead of $u(t)$.

Solving (8) as an optimization problem

DOI: 10.1002/rnc

$$\begin{array}{rl} min & u^T u \\ s.t. & v_f = B_f u \end{array} \tag{10}$$

yields

$$u_p = B_f^T (B_f B_f^T)^{-1} v_f, \tag{11}$$

where u_p is the solution of (8) using the pseudo inverse method. The pseudo inverse method is the simplest method of control allocation that gives the optimal solution and clearly is faster than the optimization-based control allocation methods. The main drawback is the fact that the control signal constraints are not considered and are easily violated. In what follows a modification of the pseudo inverse method is proposed that forces the control signals to be placed in the defined limitations. Let u_p be the optimal solution that may exceed the constraints and define a correction vector u_n such that

$$u = u_p - u_n \tag{12}$$

So the virtual control signal would be

$$v_f = B_f u = B_f(u_p - u_n) = B_f B_f^T (B_f B_f^T)^{-1} v_f - B_f u_n = v_f - B_f u_n \tag{13}$$

The aim is to either force the normalized control signals to be placed in the defined constraints or the total control effect generated by the actuators becomes equal to the virtual control vector. In this case, $B_f u_n$ must be equal to zero; thus, u_n is a vector that should lie in the null space of B_f. Define v_{free} as a design parameter to add freedom in choosing the vector u_n as

$$u_n = N_f v_{free}, \tag{14}$$

where $N_f^{m \times (m-k')}$ belongs to the null space of B_f. So correcting the control signal leads to

$$u = u_p - u_n = P_f v_f - N_f v_{free} = \left[\begin{array}{cc} P_f & N_f \end{array} \right] \left[\begin{array}{c} v_f^{n \times 1} \\ v_{free}^{(m-k') \times 1} \end{array} \right], \tag{15}$$

where P_f is the pseudo inverse of B_f. To fulfill the constraints for each control signal, normalization should be performed. First, the q^{th} element of the vector u_b is defined as follows:

$$u_{b_q} = \begin{cases} u_q^+ \;\; if \;\; u_{p_q} > 0 \\ u_q^- \;\; if \;\; u_{p_q} < 0 \end{cases}, q = 1, \ldots, m. \tag{16}$$

Suppose that $u_q^+, u_q^- \neq 0$, then the normalization is as

$$U^{-1} u = U^{-1} u_p - U^{-1} N_f v_{free}, \tag{17}$$

where the matrix U is

$$U = \left[\begin{array}{cccc} u_{b_1} & 0 & 0 & 0 \\ 0 & u_{b_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & u_{b_m} \end{array} \right] \Longrightarrow u_p^{norm} = U^{-1} u_p = \left[\begin{array}{c} \frac{u_{p_1}}{u_{b_1}} \\ \frac{u_{p_2}}{u_{b_2}} \\ \vdots \\ \frac{u_{pm}}{u_{bm}} \end{array} \right] \tag{18}$$

and there is one upper bound for all signals after normalization that is equal to one. Define the maximum magnitude of normalized control signal as

$$h = max \left\{ u_{p_q}^{norm} \;, q = 1, \ldots, m \right\}, \tag{19}$$

where $u_{p_q}^{norm}$ is the q^{th} normalized control signal using pseudo inverse and the indices of the maximum elements of u_p^{norm} are saved in the set s as

DOI: 10.1002/rnc

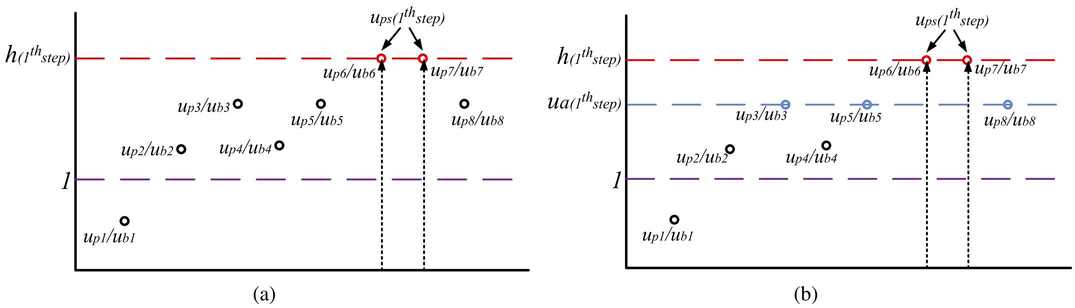

Figure 2. Schematic of selecting u_{p_s} and u_a in PAN method.

$$s = \left\{ q \middle| \frac{u_{p_q}}{u_{b_q}} = h \right\} \tag{20}$$

and $u_{p_s} \in \mathbb{R}^k$ is a vector containing u_p^{norm} elements selected according to the set s and k is equal to the number of u_p^{norm} elements, which have the same and maximum l_∞ norm or the number of members of the set s.

The main idea in using the correction vector is to place the control signals within the constraints. In the proposed method, this is achieved by using (17) as follows:

$$u^{norm} = u_p^{norm} - U^{-1} N_f v_{free} \tag{21}$$

An algorithmic approach to constraint consideration

To generate a proper u, it is necessary to calculate v_{free}. This is performed by the algorithmic approach, which is subsequently described. In this algorithm, iteration is defined as the process of allocating each virtual control vector among redundant actuators by calculating a proper u. In the proposed method, each iteration is made of some steps. In each step, the aim is to decrease the u_{p_s} elements until they are placed within the constraint at last step.

In the process of leading normalized control signals into the constraint, the proposed method uses the l_∞ norm (19) to select the normalized control signals in each step [33]. In each control allocation iteration, in every step, the l_∞ norm of the normalized control signals, the new u_{p_s}, s, and h are calculated. In each step, the control signals in u_{p_s} are decreased as follows:

$$u_a = u_{p_s} - U_s^{-1} N_{f_s} v_{free}, \tag{22}$$

where N_{f_s} is composed of the rows of the N_f matrix according to the set s, $U_s = diag(u_{b_i}), i \in s$ and u_a is an assigned value for u_{p_s} elements that should be attained in each step. It is desirable to bring u_{p_s} elements close to u_a. The u_a elements are determined according to the optimal value of u_{p_s} elements decrements that will be calculated at the end of this section.

The procedure of selecting u_{p_s} and u_a in PAN is demonstrated in Figures 2 and 3 for a system with eight actuators. In Figure 2(a) and (b), u_{p6}^{norm} and u_{p7}^{norm} are elements of u_{p_s} because they have equal and maximum l_∞ norm (h), and u_a is assumed to be known and is equal to the magnitude of $u_{p3}^{norm}, u_{p5}^{norm}$, and u_{p8}^{norm}. The 2^{nd} step is shown in Figure 3. New u_{p_s} is $u_{p_s} = [u_{p3}^{norm}, u_{p5}^{norm}, u_{p6}^{norm}, u_{p7}^{norm}, u_{p8}^{norm}]^T$ and a new u_a is assumed to be known and is equal to the magnitude of u_{p2}^{norm} and u_{p4}^{norm} . This procedure should be repeated until all signals lie in the constraints. It should be emphasized that by decreasing the u_{p_s} elements, all other control signals are free to change.

It is desired in (21) that $U^{-1} N_f v_{free}$ be minimum. From (22), the minimum value for v_{free} when N_f and N_{f_s} are assumed constant is obtained using the pseudo inverse

$$U_s^{-1} N_{f_s} v_{free} = u_{p_s} - u_a \Longrightarrow v_{free} = N_{f_s}^T \left(N_{f_s} N_{f_s}^T \right)^{-1} U_s (u_{p_s} - u_a) \tag{23}$$

Lemma 1, Lemma 2 and Remark 1 in the Appendix show that the optimal value for N_{f_s} exists. Substituting (23) in (15) yields

$$u = u_p - N_f N_{f_s}^T \left(N_{f_s} N_{f_s}^T \right)^{-1} U_s (u_{p_s} - u_a) \tag{24}$$

DOI: 10.1002/rnc

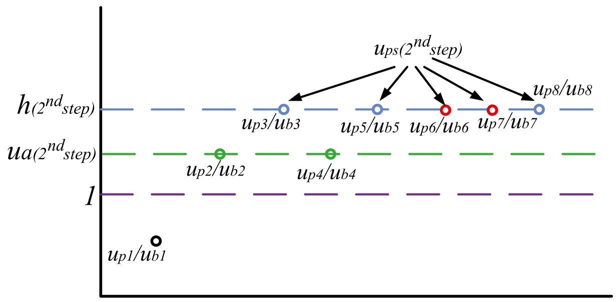

Figure 3. Schematic of control signals after one step using PAN method.

Now, the question is how to choose u_a in each step? Let $\Delta \equiv u_{p_s} - u_a$ and by the end of this section, a proper Δ is selected instead of u_a.

$$u = u_p - N_f N_{f_s}^T \left(N_{f_s} N_{f_s}^T\right)^{-1} U_s \Delta \tag{25}$$

The following ratio is always correct for the k elements of u_{p_s} in each step

$$\frac{u_{p_{si}}}{u_{b_{si}}} = \frac{u_{p_{sj}}}{u_{b_{sj}}} \quad \forall i, j \in s \tag{26}$$

It is desired to decrease the elements of u_{p_s} in each step, so the numerators of (26) should be decreased in a way that preserves the ratios in (26). Let u_{p_s} be decreased by the vector $u_a = [u_{a1}, \ldots, u_{ak}]^T$ as

$$\frac{u_{p_{si}} - u_{ai}}{u_{b_{si}}} = \frac{u_{p_{sj}} - u_{aj}}{u_{b_{sj}}} \Longrightarrow \frac{\Delta_i}{u_{b_{si}}} = \frac{\Delta_j}{u_{b_{sj}}} \Longrightarrow \Delta_j = \frac{u_{b_{sj}}}{u_{b_{si}}}\Delta_i, \quad \forall i, j \in s \tag{27}$$

Equation (27) implies that once Δ_i for an $i \in s$ is known, the other elements of vector Δ are completely determined. Assuming $|u_{aj}| < |u_{p_{sj}}|$, the sign of Δ_j is equal to the signs of $u_{p_{sj}}$ and $u_{b_{sj}}$. Then, the magnitude of Δ_j is calculated and its sign would be the sign of $u_{b_{sj}}$.

$$\begin{cases} \Delta_j > 0 \; for \;\; u_{p_{sj}} > 0 \\ \Delta_j < 0 \; for \;\; u_{p_{sj}} < 0 \end{cases}, \quad \forall j \in s. \tag{28}$$

Using (27) and (28) yields

$$\begin{cases} for \;\; \Delta_i > 0 \Longrightarrow \Delta_j = \frac{u_{b_{sj}}}{u_{b_{si}}}|\Delta_i| \\ for \;\; \Delta_i < 0 \Longrightarrow \Delta_j = \frac{u_{b_{sj}}}{u_{b_{si}}}(-|\Delta_i|) \end{cases} \quad \forall i, j \in s\,. \tag{29}$$

For simplicity, let $\bar{\Delta} = |\Delta_i|$ for a selected $i \in s$. Then for the selected i, (27) is rewritten as follows:

$$\Delta_j = \frac{u_{b_{sj}}}{|u_{b_{si}}|}\bar{\Delta}\,, \quad \forall j \in s. \tag{30}$$

Defining u_r for the selected i as follows:

$$u_r \equiv N_f N_{f_s}^T (N_{f_s} N_{f_s}^T)^{-1} U_s \frac{u_{b_s}}{|u_{b_{si}}|}. \tag{31}$$

Using (25), (30), and (31) yields

$$u = u_p - u_r \bar{\Delta}. \tag{32}$$

The magnitude of $\bar{\Delta}$ is still unknown. There are multiple choices for $\bar{\Delta}$ in order to decrease u_{p_s} elements. Each control allocation methodology uses special procedure to handle the actuator constraints, for example, direct allocation [8] and redistributed pseudo inverse [5] use different algorithms to achieve this goal. The proposed $\bar{\Delta}$ for PAN manipulates u_p^{norm} elements in a way that u_{p_s} elements equal to $u_{p_\ell}^{norm}$ for an $\ell, \ell = 1, \ldots, m, \ell \notin s$.

DOI: 10.1002/rnc

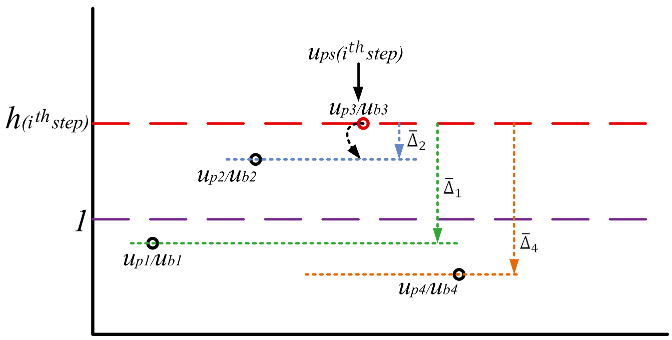

Figure 4. Choosing the smallest $\bar{\Delta}_\ell$ in order to decrease control signal variations.

Representing $\bar{\Delta}$ for each $u_{p_\ell}^{norm}, \ell = 1, \ldots, m, \ell \notin s$ by $\bar{\Delta}_\ell$, the following equation should be satisfied to handle the actuator constraints in PAN

$$\frac{u_{p_{si}} - u_{r_{si}}\bar{\Delta}_\ell}{u_{b_{si}}} = \frac{u_{p_\ell} - u_{r_\ell}\bar{\Delta}_\ell}{u_{b_\ell}}, \quad \forall i \in s\ , \ \ \ell = 1, \ldots, m, \ \ \ell \notin s. \tag{33}$$

Using (33) yields

$$\bar{\Delta}_\ell = \frac{\frac{u_{p_{si}}}{u_{b_{si}}} - \frac{u_{p_\ell}}{u_{b_\ell}}}{\frac{u_{r_{si}}}{u_{b_{si}}} - \frac{u_{r_\ell}}{u_{b_\ell}}}, \quad \forall i \in s\ , \ \ \ell = 1, \ldots, m, \ \ \ell \notin s. \tag{34}$$

Equation (34) determines $m - k$ choices for $\bar{\Delta}$, (i.e., $\bar{\Delta}_1, \ldots, \bar{\Delta}_\ell, \ell = 1, \ldots, m, \ \ell \notin s$.). In order to apply minimal changes, the minimum of $\bar{\Delta}_\ell$ should be selected as follows:

$$\bar{\bar{\Delta}} = min\{\bar{\Delta}_\ell | \ell = 1, \ldots, m\ , \ \ \ell \notin s\} \tag{35}$$

Figure 4 illustrates the procedure of selecting $\bar{\Delta}_\ell$ for a system with four actuators. In this system, u_{p_s} has one element that is $u_{p_3}^{norm}$. For $u_{p_1}^{norm}$, $u_{p_2}^{norm}$, and $u_{p_4}^{norm}$, $\bar{\Delta}_\ell, \ell = 1, \ldots, m, \ \ell \notin s$ are computed. There are three $\bar{\Delta}_\ell, \ell = 1, \ldots, m, \ \ell \notin s$ that are shown by three straight downward arrows. By using minimum $\bar{\Delta}_\ell, \ell = 1, \ldots, m, \ \ell \notin s$ (i.e., $\bar{\Delta}_2$), it can be seen that $u_{p_3}^{norm}$ is decreased by the curved arrow.

Directionality is an important characteristic of control allocation methods that preserves the direction of each control signal. Some applications like flight control requires this property [9]. Pseudo inverse is the basic solution for allocating control signals but cannot consider the constraints. In order to place control signals into the constraints, the direction of the control signals may change. In the proposed method, it is possible that during the reduction of u_{p_s} elements, the sign of the numerator in (33) changes, and it consequently changes the direction of control signals. To solve this problem, define d_ℓ as

$$d_\ell = sign(u_{p_\ell}) \times sign(u_{r_\ell}), \quad \ell = 1, \ldots, m\ , \ \ \ell \notin s. \tag{36}$$

If d_ℓ is positive, there is a possibility that the sign of $u_{p_\ell} - u_{r_\ell}\bar{\Delta}_\ell, \ell = 1, \ldots, m, \ \ell \notin s$ changes, and consequently $u_{b_\ell}, \ell = 1, \ldots, m, \ell \notin s$ must change. In order to solve this problem, $\bar{u}_{b_q}, q = 1, \ldots, m$ and $\hat{\Delta}_\ell\ , \ell = 1, \ldots, m, \ \ell \notin s$ are defined as follows:

$$\bar{u}_{b_q} = \begin{cases} u_q^+ \ if \ u_{p_q} < 0 \\ u_q^- \ if \ u_{p_q} > 0 \end{cases}, \ q = 1, \ldots, m. \tag{37}$$

$$\hat{\Delta}_\ell = \frac{\frac{u_{p_{si}}}{u_{b_{si}}} - \frac{u_{p_\ell}}{\bar{u}_{b_\ell}}}{\frac{u_{r_{si}}}{u_{b_{si}}} - \frac{u_{r_\ell}}{\bar{u}_{b_\ell}}}, \quad \forall i \in s\ , \ \ \ell = 1, \ldots, m, \ \ \ell \notin s. \tag{38}$$

DOI: 10.1002/rnc

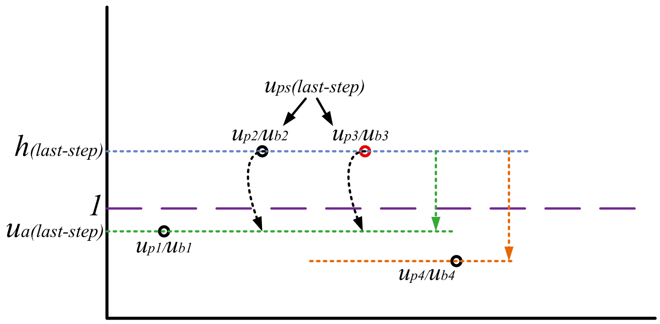

Figure 5. Last step that should be modified.

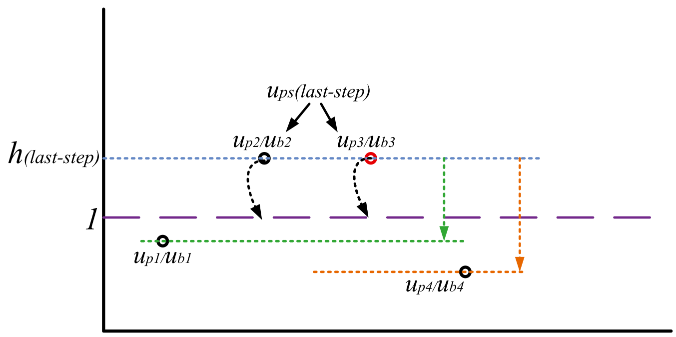

Figure 6. Last step of the PAN method.

Finally, with directionality consideration, equation (35) is written as

$$\bar{\bar{\Delta}} = min\{min(\hat{\Delta}_\ell, \bar{\Delta}_\ell)|\ell = 1, \ldots, m\ ,\ \ \ell \notin s\} \tag{39}$$

Using (32) and (39) yields

$$u = u_p - u_r\bar{\bar{\Delta}}. \tag{40}$$

At the last step, by repeating the previous procedure, the normalized control signals satisfy the limitations as shown in Figure 5, but this is suboptimal. To improve the responses, as soon as $u_{s_i}/u_{b_{si}} < 1, \forall i \in s$, instead of reducing the u_{p_s} elements to achieve u_a that it is shown in Figure 5, set the u_{p_s} elements equal to the constraint that is shown in Figure 6. This part is added to the algorithm in order to use the maximum capacity of the actuators in addition to handling actuators limitations. This is performed by using () at the last step of the algorithm.

$$\bar{\bar{\Delta}} = |u_{p_{si}} - 1|,\ \ \forall i \in s. \tag{41}$$

3.2. The infeasible solution case

The design vector v_{free} has $m - k'$ elements so it can just control $m - k'$ control signals. If u_{p_s} has more than $m - k'$ elements, the proposed method cannot force all of them into their constraints. In this case, the problem is called *infeasible*. The proposed method has maximum $m - k'$ steps so if after the $(m - k')^{th}$ step, the signals do not satisfy their constraints, it is an infeasible problem. A solution with low computational complexity for this problem is provided by the simple direct allocation method [8]. Let

$$\alpha = max\left\{u_{p_q}^{norm}|q = 1, \ldots, m\right\} \tag{42}$$

then decrease all the signals in a way that the maximum signals lie on the border value. That is,

DOI: 10.1002/rnc

$$u_{final} = u_p^{norm}/\alpha. \tag{43}$$

Note that the vector u_p^{norm} in (43) and its elements in (42) are composed of the normalized control signals after $(m-k')^{th}$ step.

3.3. Singularity avoidance

In the proposed control allocation method, if an actuator completely fails ($k_i = 1$), the $N_{f_s} N_{f_s}^T$ may be singular. In this case, using singularity avoidance [34] solves the problem. The singularity occurrence is dependent on the members of the set s, because the rows of N_{f_s} are chosen from N_f based on the members of s in each step.

Please note that in the case of singularity in $N_{f_s} N_{f_s}^T$, the PAN steps should be stopped in order to avoid inaccurate solutions and the control allocation algorithm is completed by solving the following optimization problem using the slack variable ε [27, 34]

$$\begin{gathered} u(t) = \underset{\varepsilon,u}{argmin} \left(||\varepsilon||^2_{Q_\varepsilon} + ||u||^2_{R_u}\right) \\ \varepsilon = v_f - B_f u, \quad u \in \Omega \end{gathered}. \tag{44}$$

However, if the complete actuator failure does not result in the singularity of $N_{f_s} N_{f_s}^T$, the resorted optimization problem is no longer necessary and the proposed procedure is continued.

The term $||\varepsilon||^2_{Q_\varepsilon}$ penalizes the error between the virtual control and the achieved control signal. The diagonal weights in the matrix $Q_\varepsilon > 0$ are chosen so large that $\varepsilon \approx 0$ whenever possible. This method is not related to the pseudo inverse method, but its simplicity and efficiency is the reason for its selection.

The complete flow chart of PAN

The complete procedure of the PAN algorithm in order to allocate control signals among faulty redundant actuators is demonstrated in Figure 7. Each iteration starts from the following initial data: u^+, u^-, v, B, W. The information about u^+ and u^- are determined using actuators' data sheets. v is provided by a stabilizing controller, and B is supposed to be known as the control matrix. Also, W shows the effects of actuator faults and should be identified using a fault identification algorithm. These initial values produce u_p and u_b in each iteration.

If the u_p elements are not in their constraint limits, the proposed algorithm is activated and as soon as $h \leqslant 1$, the algorithm stops and the proper u is calculated. If the step counter exceeds the $m-k'$, the simple direct allocation method leads the infeasible solution into the constraint. Also, in order to avoid singularity that may occur in some steps, an optimization problem should be solved.

4. EXAMPLES

In order to have a control allocation method that adaptively tolerates faults, it is necessary to have an identification method. In the following examples, a Recursive Least Square algorithm (RLS) [27] has been used.

4.1. Linear unstable scalar model

Consider a linear state space model [27] of a faulty system (6) where $A = 1.2$, $B = [1 \;\; 1 \;\; 1]$ and $W(t)$ is as below

$$W(t) = \begin{cases} diag(1,1,1) & for\ t < 180(s) \\ diag(1,1,0) & for\ 180(s) \leqslant t \leqslant 320(s) \\ diag(1,0.5,0) & for\ t \geqslant 320(s) \end{cases} \tag{45}$$

There is a complete failure in the third actuator at $t = 180(s)$ and a 50% loss of effectiveness fault in the second actuator at $t = 320(s)$. The system has redundancy in actuators, and the control signals have constraints as

$$-5 \leqslant u_i(t) \leqslant 5\,, \quad i = 1,2,3. \tag{46}$$

DOI: 10.1002/rnc

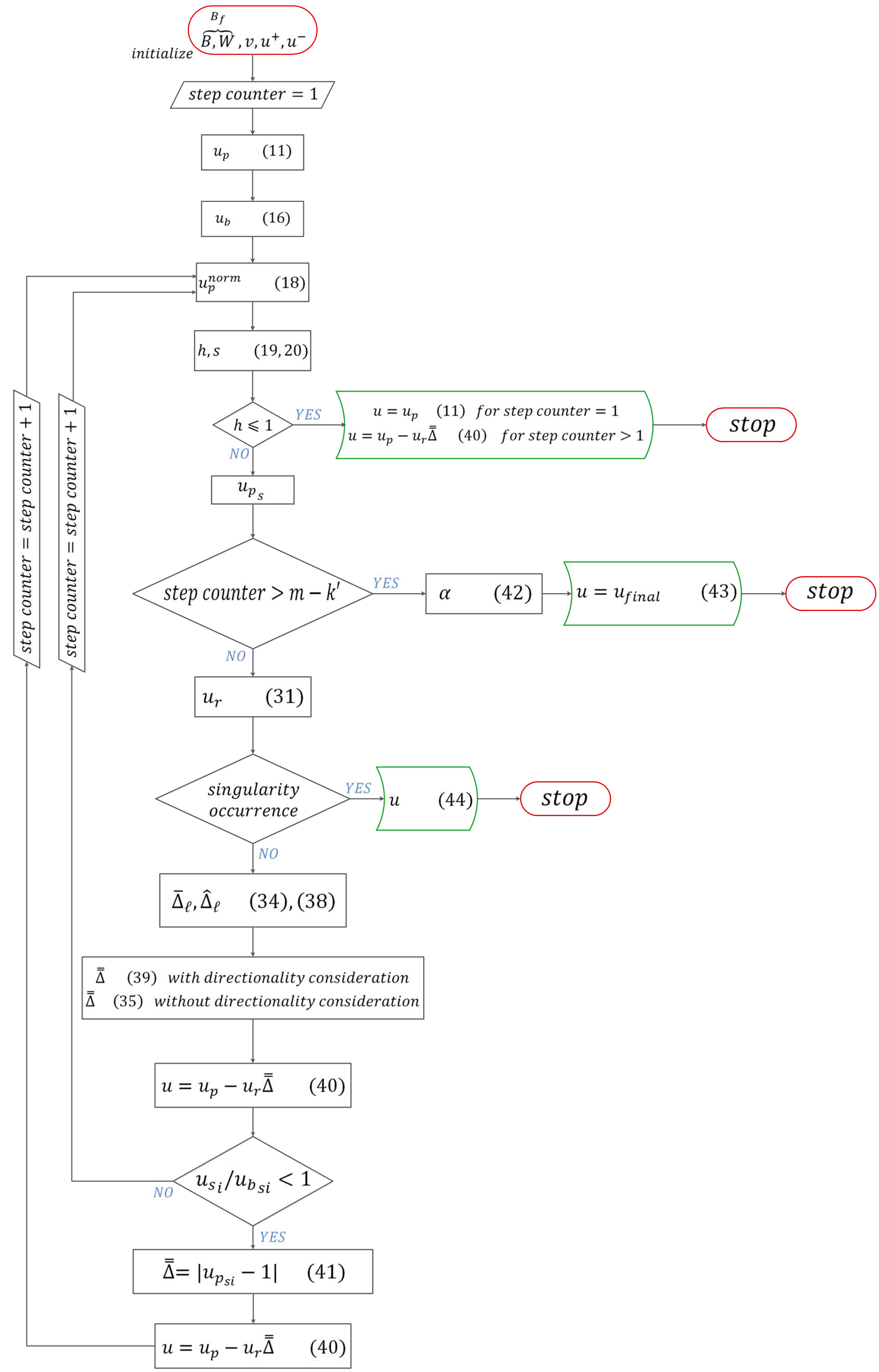

Figure 7. The flow chart of PAN.

DOI: 10.1002/rnc

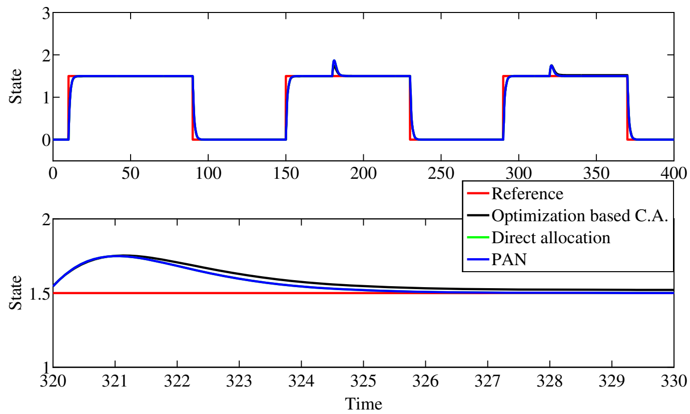

Figure 8. System's state using three control allocation methods in faulty conditions.

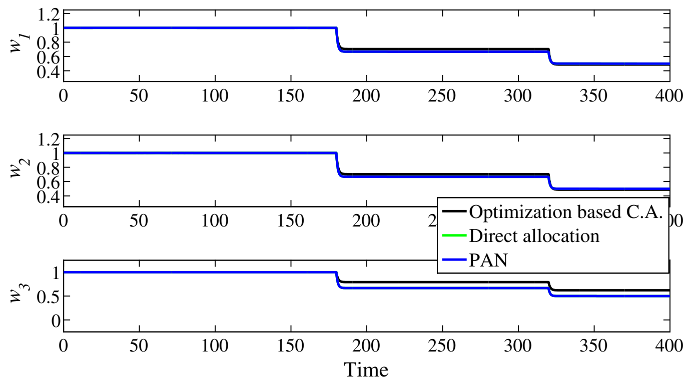

Figure 9. Fault identification using RLS.

The virtual control signal is built by gain −2.2 in the feedback.

Figure 8 shows the effect of using the FTC. Faults are tolerated, and only small jumps are present at the time that fault occurs. It is also obvious in Figure 8 that the optimization-based control allocation [27] is not as accurate as the two other methods. The response of direct allocation method [8, 9] and PAN are similar as shown in Figure 9. The diagonal elements of the effectiveness matrix (w_1, w_2, w_3) are identified in Figure 9. By increasing the persistent excitation (PE) degree of signals, it is seen in Figure 10 that the estimation of effectiveness coefficients are improved. Figure 11 also shows the system states after increasing PE degree of signals. It is seen that the FTC using PAN tolerates the fault better than the other two methods.

4.2. *An over-actuated supply vessel*

Consider the model of a supply vessel given in [27, 35]. The systems position is represented by $\eta = [x, y, \phi]^T$ where x, y are the earth-fixed positions and ϕ is the yaw angle, as illustrated in Figure 12. The body-fixed velocities are also represented by $\upsilon = [\nu, u, r]^T$, where ν is the forward velocity, u is the lateral velocity, and r is the yaw angular velocity. To normalize the variables, the following Bis-scaling change of variables is accomplished [36]

$$\begin{aligned} \eta &= diag(L, L, 1)\eta'' \\ \upsilon &= diag(\sqrt{gL}, \sqrt{gL}, \sqrt{\frac{g}{L}})\upsilon'', \\ t &= \sqrt{L/g}t'' \end{aligned} \quad (47)$$

DOI: 10.1002/rnc

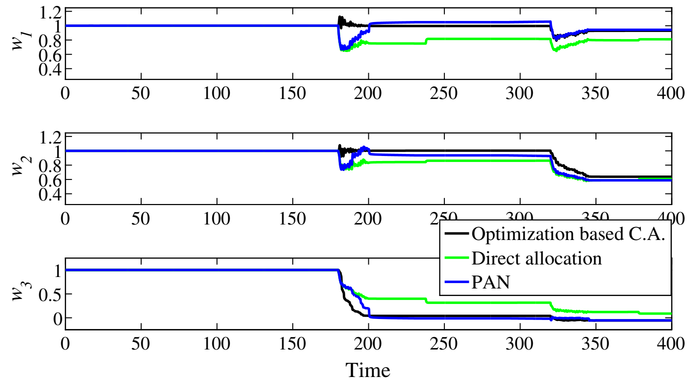

Figure 10. Fault identification using RLS, increasing PE of signals.

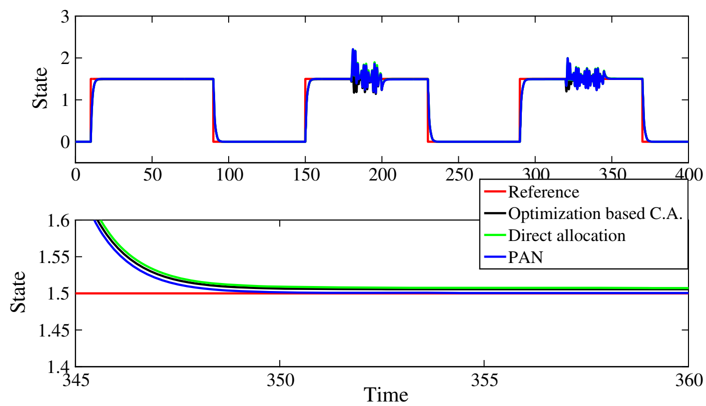

Figure 11. System's state using three control allocation methods in faulty conditions, increasing PE of signals.

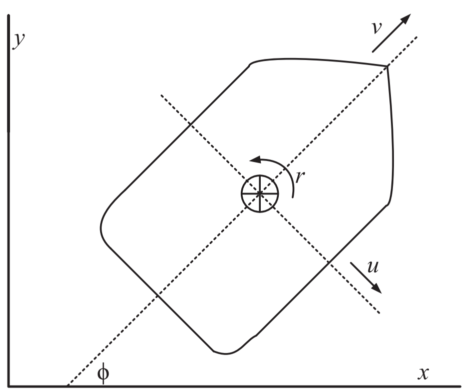

Figure 12. Schematic of a ship.

where $L = 76.2(m)$ is the supply vessel length and g is the gravity acceleration. Also, $m = 6.4 \times 10^6(kg)$ is the supply vessel mass. The nonlinear model of the system is given as

$$\begin{aligned} \dot{\eta}''(t'') &= J(\eta''(t''))\upsilon''(t'') \\ M\dot{\upsilon}''(t'') + C(\upsilon'')\upsilon''(t'') + D\upsilon''(t'') &= BW(t'')u(t'') + k_b J^T(\eta''(t''))b(t'') \end{aligned}. \tag{48}$$

Table I. The system matrices.

M	Inertia matrix
C	Coriolis/sentripetal matrix
D	Hydrodynamic damping matrix
J	Rotation matrix around the yaw axis

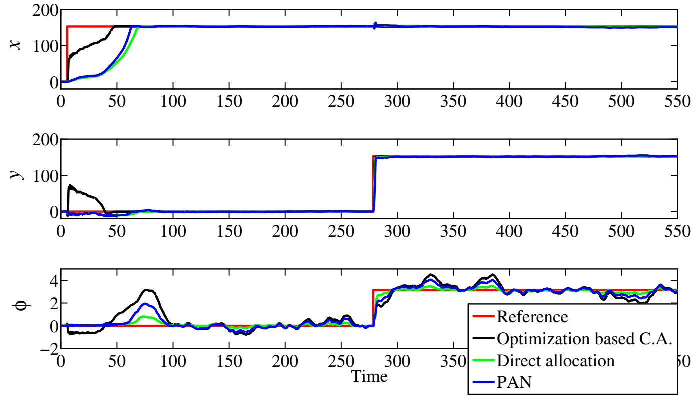

Figure 13. System's states using the three control allocation methods in faulty conditions.

The part $k_b J^T(\eta''(t''))b(t'')$ is added to the model as the effect of disturbance (ocean currents) [27, 36]. So the disturbance differential equation is

$$\dot{b} = -T_b^{-1} b + E_b q_b, \tag{49}$$

where b is referred to as the bias vector, q_b is a zero mean Gaussian white noise, T_b is a diagonal matrix containing the bias time constant, E_b is a diagonal matrix acting as a scaling factor for q_b.

$$T_b^{-1} = E_b = 10^{-3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad k_b = 0.2 \tag{50}$$

The system matrices are defined in Table I [35].

Where

$$M = \begin{bmatrix} 1.1274 & 0 & 0 \\ 0 & 1.8902 & -0.0744 \\ 0 & -0.0744 & 0.1278 \end{bmatrix}, \tag{51}$$

$$C = \begin{bmatrix} 0 & 0 & -1.8902v'' + 0.0744r'' \\ 0 & 0 & 1.1274u'' \\ 1.8902v'' - 0.0744r'' & -1.1274u'' & 0 \end{bmatrix}, \tag{52}$$

$$D = \begin{bmatrix} 0.0414 & 0 & 0 \\ 0 & 0.1775 & -0.0141 \\ 0 & -0.1073 & 0.0568 \end{bmatrix}, \tag{53}$$

$$J(\eta'') = \begin{bmatrix} cos(\phi'') & -sin(\phi'') & 0 \\ sin(\phi'') & cos(\phi'') & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{54}$$

DOI: 10.1002/rnc

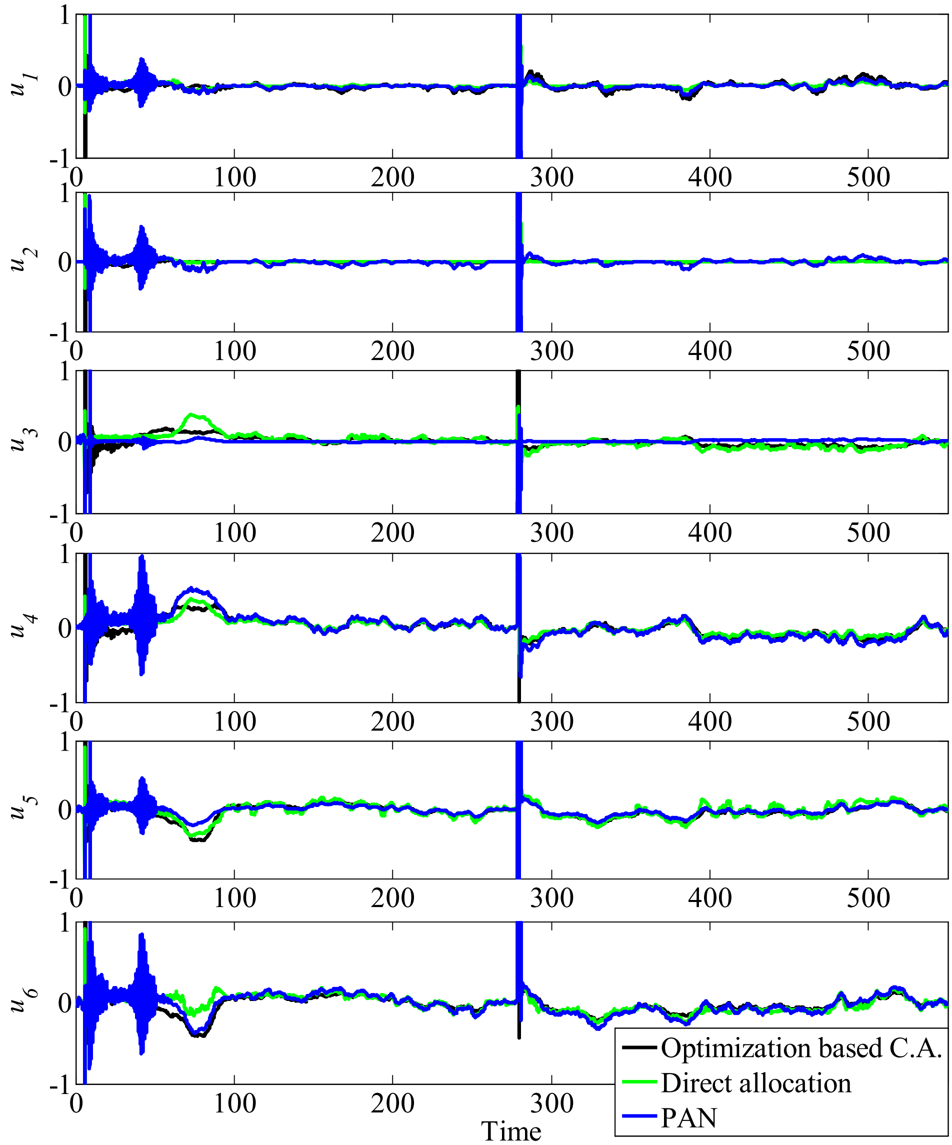

Figure 14. Six constrained control signals of the system.

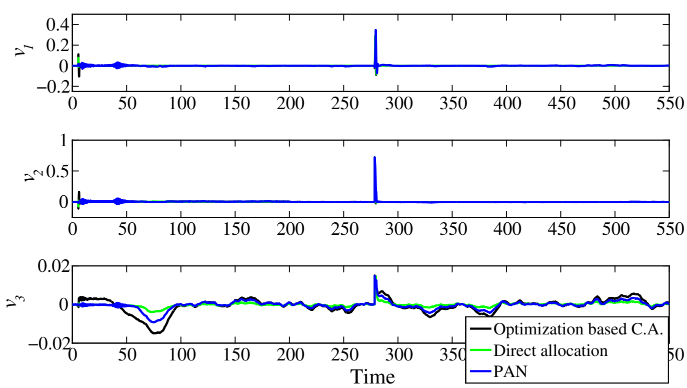

Figure 15. Virtual control signals.

DOI: 10.1002/rnc

Table II. Comparison between computational costs of the three control allocation methods

Optimization Based Direct Allocation	0.0111(s)
Optimization Based Control Allocation	0.0104(s)
PAN	0.5804(ms)

$$B = 10^{-3}\begin{bmatrix} 13 & 13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11.6 & 11.6 & 6 & 6.7 \\ 0 & 0 & -4.6 & -4.6 & 2.7 & 2.2 \end{bmatrix}. \quad (55)$$

The actuator faults are assumed as

$$W(t) = \begin{cases} diag(1,1,1,1,1,1) & for\ t < 61.3(s) \\ diag(1,0,0.5,1,1,1) & for\ t \geqslant 61.3(s) \end{cases} \quad (56)$$

Also, $u(t) = [u_1(t), \ldots, u_6(t)]^T$ are the input signals to each actuator where $u_1(t), u_2(t)$ are the signals for the two identical main propellers and $u_3(t), \ldots, u_6(t)$ are the control signals for the transverse thrusters.

Using a constant control law to produce the virtual control signal

$$V = K_\upsilon[J^T(\eta)K_\eta(\eta_d - \eta) - \upsilon] \quad (57)$$

The controller gains are $K_\upsilon = 0.75 I_{3\times3}$, $K_\eta = 0.063 I_{3\times3}$ and the sampling time for both the control law and the control allocator is $T_s = 0.2785(s)$. Also, η_d is the desired reference trajectory as [27]

$$\eta_d(t) = \begin{cases} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T & for\ t < 5.57(s) \\ \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T & for\ 5.57(s) \leqslant t \leqslant 278.5(s) \\ \begin{bmatrix} 2 & 2 & \pi \end{bmatrix}^T & for\ t > 278.5(s) \end{cases} \quad (58)$$

Figure 13 shows the states of the faulty system, which track the references using the three control allocation methods. Although the tracking of x in the first 50 s is better by using the optimization based control allocation, but y and ϕ are tracked better in the first fifty seconds using the PAN method. The oscillations of signals are due to ocean current disturbance. Figure 14 shows the control signals, and Figure 15 shows the virtual-control signals of the system. It can be seen that all control signals lie in their constraints.

An advantage of the proposed method is its low computational operation that makes it very fast. In Table II, the average time for each method is given. Direct allocation that is based on optimization is the slowest method, and the PAN method is the fastest. The data in Table II is the average time of 4000 iterations. The results were obtained on a 64-bit Windows 7, with Intel Core 2 Duo CPU. The algorithms were implemented as m-files in MATLAB R2010a.

5. CONCLUSIONS

A fault-tolerant controller using adaptive control allocation based on the pseudo inverse along the null space of the control matrix is proposed in this paper. Adaptation of the proposed control allocation method using the RLS technique is presented. The infeasibility problem is solved, and a method for singularity avoidance is given. The main property of the proposed methodology is its low computational cost with optimal solution. The simulation results of the two examples show the effectiveness of the proposed method in comparison with the optimization-based control allocation and the direct allocation strategies.

DOI: 10.1002/rnc

APPENDIX

Lemma 1
Consider a matrix $A^{(n\times m)}$ and its gain α that lies in $\underline{\alpha} \leqslant \alpha \leqslant \overline{\alpha}$. By omitting one or more rows of A, the gain of the new matrix is β that lies in $\underline{\beta} \leqslant \beta \leqslant \overline{\beta}$. Then the following inequality holds

$$\beta \leqslant \alpha \tag{A.1}$$

Proof
Consider the equality below

$$r^{n\times 1} = A^{n\times m} x^{m\times 1} \tag{A.2}$$

Using l_2 norm of (A.2)

$$||r|| = ||Ax|| \Longrightarrow ||r|| = \alpha||x|| \tag{A.3}$$

By omitting one row, (A.2) can be written as

$$r'^{(n-1)\times 1} = A'^{(n-1)\times m} x^{m\times 1} \tag{A.4}$$

Using l_2 norm of (A.2)

$$||r'|| = ||A'x|| \Longrightarrow ||r'|| = \beta||x|| \tag{A.5}$$

Now compare (A.3) and (A.5)

$$||r'|| \leqslant ||r|| \Longrightarrow \beta \leqslant \alpha \tag{A.6}$$

□

Lemma 2
To minimize $||N_f v_{free}||$, where N_f is an unknown constant matrix, the maximum singular value of N_f is one.

Proof
Using l_2 norm of $N_{f_s} v_{free} = u_{p_s} - u_a$ yields

$$\|N_{f_s} v_{free}\| = \|u_{p_s} - u_a\| \Longrightarrow \sigma'\|v_{free}\| = \|u_{p_s} - u_a\| \Longrightarrow \|v_{free}\| = \frac{1}{\sigma'}\|u_{p_s} - u_a\| \tag{A.7}$$

Let σ and σ' be the matrix gains of N_f and N_{f_s} respectively and $\underline{\sigma'} \leqslant \sigma' \leqslant \overline{\sigma'}$ and $\underline{\sigma} \leqslant \sigma \leqslant \overline{\sigma}$, then

$$\|N_f v_{free}\| = \sigma\|v_{free}\| \Longrightarrow \|N_f v_{free}\| = \frac{\sigma}{\sigma'}\|u_{p_s} - u_a\| \tag{A.8}$$

By assuming $\|u_{p_s} - u_a\|$ to be constant, to minimize $\|N_{f_s} v_{free}\|$, $\frac{\sigma}{\sigma'}$ should be minimized. Using *Lemma 1* results that $\frac{\overline{\sigma}}{\underline{\sigma'}}$ should be equal to one. □

Remark 1
Using the singular value decomposition for a matrix $H^{n\times m}$ as

$$H = RSQ \tag{A.9}$$

where $RR^T = R^T R = I_n$ and $QQ^T = Q^T Q = I_m$. If $rank(H) = r$, then $(m-r)$ last columns of Q make an orthogonal base for the null space of H defined as $N^{m\times(m-r)}$. Then the singular values of N are as

$$\sigma_i(N) = \sqrt{\lambda_i(N^T N)} = \sqrt{\lambda_i(I_{m-r})} = 1, \quad i = 1, \dots, m-r \tag{A.10}$$

DOI: 10.1002/rnc

REFERENCES

1. Edwards C, Lombaerts T, Smaili H. *Fault Tolerant Flight Control: A Benchmark Challenge*. Springer: Berlin, 2010.
2. Frank PM. Fault diagnosis in systems using analytical and knowledge-based redundancy–a survey and some new results. *Automatica* 1990; **26**:459–474.
3. Yiu YK, Meng J, Li ZX. Auto-calibration for a parallel manipulator with sensor redundancy. *International Conference on Robotics and Automation*, Taipei, Taiwan, 2003; 3660–3665.
4. Johansen TA, Fossen TI. Control allocation-a survey. *Automatica* 2013; **49**:1087–1103.
5. Bordignon KA. Constrained control allocation for systems with redundant control effectors. *PHD thesis*, Department of Aerospace and Ocean Engineering, Virginia Polytechnic Institute and State University, 1996.
6. Enns D. Control allocation approaches. *Proceeding of the AIAA Guidance. Control and Dynamics Conference*, Boston, MA, USA, 1998; 98–108.
7. Buffington JM, Enns DF. Lyapunov stability analysis of daisy chain control allocation. *Journal of Guidance, Control, and Dynamics* 1996; **19**:1226–1230.
8. Durham W. Constrained control allocation. *Journal of Guidance, Control, and Dynamics* 1993; **16**:717–725.
9. Bodson M. Evaluation of optimization methods for control allocation. *Journal of Guidance, Control, and Dynamics* 2002; **25**:703–711.
10. Bodson M, Frost SA. Control allocation with load balancing. *Journal of Guidance, Control, and Dynamics* 2011; **34**:1017–1027.
11. Petersen JAM, Bodson M. Constrained quadratic programming techniques for control allocation. *IEEE Transaction on Control Systems Technology* 2006; **14**:91–98.
12. Harkegard O. Efficient active set algorithms for solving constrained least squares problems in aircraft control allocation. *In Proceeding IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 2002; 1295–1300.
13. Harkegard O, Torkel Glad S. Resolving actuator redundancy-optimal control vs. control allocation. *Automatica* 2005; **41**:137–144.
14. Johansen TA. Optimizing nonlinear control allocation. *IEEE Conference on Decision and Control*, Nassau, Bahamas, 2004; 3435–3440.
15. Taghirad HD, Bedoustani YB. An analytic-iterative redundancy resolution scheme for cable-driven redundant parallel manipulators. *IEEE Transaction on Robotics* 2011; **27**:1137–1143.
16. Zaccarian L. Dynamic allocation for input redundant control systems. *Automatica* 2009; **45**:1431–1438.
17. Cristofaro A, Galeani S. Output invisible control allocation with steady-state input optimization for weakly redundant plants. *IEEE Conference on Decision and Control*, Los Angeles, CA, USA, 2014; 4246–4253.
18. Serrani A. Output regulation for over-actuated linear systems via inverse model allocation. *IEEE Conference on Decision and Control*, Maui, HI, USA, 2012; 4871–4876.
19. Zhang TA, Jiang J. Bibliographical review on reconfigurable fault-tolerant control systems. *Annual Reviews in Control* 2008; **32**:229–252.
20. Mahmoud MM, Jiang J, Zhang Y. *Active Fault Tolerant Control Systems: Stochastic Analysis and Synthesis*. Springer: Berlin, 2003.
21. Zhao Q, Jiang J. Reliable state feedback control system design against actuator failures. *Automatica* 1998; **34**: 1267–1272.
22. Yang GH, Wang JL, Soh YC. Reliable H_∞ controller design for linear systems. *Automatica* 2001; **37**:717–725.
23. Liao F, Wang JL, Yang GH. Reliable robust flight tracking control: an LMI approach. *IEEE Transaction on Control Systems Technology* 2002; **10**:76–89.
24. Ye D, Yang GH. Adaptive fault-tolerant tracking control against actuator faults with application to flight control. *IEEE Transaction on Control Systems Technology* 2006; **14**:1088–1096.
25. Alwi H, Edwards C. Fault tolerant control using sliding mode with on-line control allocation. *Automatica* 2008; **44**:1859–1866.
26. Liu Y, Crespo LG. Adaptive control allocation in the presence of actuator failures. *Journal of Control Science and Engineering* 2012; **2012**:1–16.
27. Casavola A, Garone E. Fault-tolerant adaptive control allocation schemes for overactuated systems. *International Journal of Robust and Nonlinear Control* 2001; **37**:717–725.
28. Tjonnas J, Johansen TA. Adaptive control allocation. *Automatica* 2008; **44**:2754–2765.
29. Napolitano MR, Song Y, Seanor BA. On-line parameter estimation for restructurable flight control systems. *Aircraft Design* 2001; **4**:19–50.
30. Cristofaro A, Johansen TA. Fault tolerant control allocation using unknown input observers. *Automatica* 2014; **50**:1891–1897.
31. Tao G, Joshi SM, Ma X. Adaptive state feedback and tracking control of systems with actuator failures. *IEEE Transactions on Automatic Control* 2001; **46**:78–95.
32. Bozorgnia D, Khaki Sedigh A. A new control allocation methodology based on the pseudo inverse along the null space. *21th Iranian Conference on Electrical Engineering*, Mashhad, Iran, 2013; 1–4.
33. Frost SA, Taylor BR, Jutte CV, Burken JJ, Trinh KV, Bodson M. A framework for optimal control allocation with structural load constraints. *AIAA Atmospheric Flight Mechanics Conference*, Toronto, Canada, 2010.
34. Johansen TA, Fossen TI, Berge SP. Constrained nonlinear control allocation with singularity avoidance using sequential quadratic programming. *IEEE Transaction on Control Systems Technology* 2004; **12**:211–216.
35. Fossen TI, Berge SP. Nonlinear vectorial backstepping design for global exponential tracking of marine vessels in the presence of actuator dynamics. *36th Conference on Decision and Control*, San Diego, CA, USA, 1997; 4237–4242.
36. Fossen TI. *Guidance and Control of Ocean Vehicles*. John Wiley & Sons: Chichester, 1994.

DOI: 10.1002/rnc